OLIMPIADA NATIONALA DE MATEMATICA ETAPA FINALA - 2016

1.

Fie $n \in \mathbb{N}, n \geq 2$ si numerele reale $a_1, a_2, ..., a_n \in (0, \infty)$ astfel incat $a_1 \cdot a_2 \cdot ..... \cdot a_n = 1$. Demonstrati ca functia

$$f : (0, \infty) \to \mathbb{R},\ f(x) = \left(1 + a_1^x\right)\left(1 + a_2^x\right)......\left(1 + a_n^x\right)$$

este crescatoare.

Solutie.

Efectuand inmultirile se obtine:

$$f(x) = 1 + \sum_{1 \leq i \leq n} a_i^x + \sum_{1 \leq i < j \leq n} a_i^x a_j^x + \sum_{1 \leq i < j < k \leq n} a_i^x a_j^x a_k^x ...... + a_1^x a_2^x ... a_n^x \quad (1).$$

Deoarece $a_1 \cdot a_2 \cdot ..... \cdot a_n = 1$, expresia functiei f din (1) se poate scrie ca o suma de expresii de forma $a^x + \frac{1}{a^x}$, unde $a \in (0, \infty)$ (2).

Considerand functia $g : (0, \infty) \to \mathbb{R},\ g(x) = a^x + \frac{1}{a^x}$, pentru $x, t > 0$ avem:

$$g(x+t) - g(x) = \frac{a^{2x+2t} + 1}{a^{x+t}} - \frac{a^{2x} + 1}{a^x} =$$

$$\frac{a^{2x+2t} - a^{2x+t} + 1 - a^t}{a^{x+t}} = \frac{\left(a^{2x+t} - 1\right)\left(a^t - 1\right)}{a^{x+t}} \overset{x,t>0}{\geq} 0,$$

de unde rezulta g crescatoare (3).

Atunci din $(1), (2)$ si (3) se obtine f crescatoare.

2.

Fie $f : \mathbb{R} \to \mathbb{R}$ o functie cu proprietatile:

$$(P1)\ f(x+y) \leq f(x) + f(y),$$

$$(P2)\ f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y),$$

pentru orice $x, y \in R$ si $t \in [0,1]$.

a. Demonstrati ca oricare ar fi $a \leq b \leq c \leq d$, astfel incat $d - c = b - a$, are loc inegalitatea $f(b) + f(c) \leq f(a) + f(d)$.

b. Demonstrati ca

$$f\left(x_1 + ... + x_{n-1} + x_n\right) + (n-2)\left[f\left(x_1\right) + ... + f\left(x_{n-1}\right) + f\left(x_n\right)\right] \geq \sum_{1 \leq i < j \leq n} f\left(x_i + x_j\right),$$

pentru orice $n \in \mathbb{N}, n \geq 3$ si $x_1, x_2, ..., x_n \in \mathbb{R}$.

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Solutie.

$a) Fie\ t \in [0,1]\ a.i.\ b = ta + (1-t)d\ si\ c = (1-t)a + td \Rightarrow \begin{cases} f(b) \overset{ip.}{\leq} tf(a) + (1-t)f(d) \\ f(c) \overset{ip}{\leq} (1-t)f(a) + tf(d) \end{cases}$

$\Rightarrow f(b) + f(c) \leq f(a) + f(d).$

$b)$ Prin inductie folosind urmatoarea schema de rezolvare:

I. $P(3)$ – adevarat;

II. $f(x_1 + ... + x_n + x_{n+1}) + (n-2)[f(x_1) + ... + f(x_{n-1}) + f(x_n + x_{n+1})] \overset{P(n)A}{\geq}$

$\sum_{1 \leq i < j \leq n-1} f(x_i + x_j) + \sum_{1 \leq i \leq n-1} f(x_i + x_n + x_{n+1})$; **III.** $\sum_{1 \leq i \leq n-1} f(x_i + x_n + x_{n+1}) \geq$

$\sum_{1 \leq i \leq n-1} [f(x_i + x_n) + f(x_i + x_{n+1}) - f(x_i)] + (n-1)[f(x_n + x_{n+1}) - f(x_n) - f(x_{n+1})]$;

IV. $P(n) \Rightarrow P(n+1)$;

Intr-adevar:

I. Fie $x, y, z \in \mathbb{R}$ si $x \geq y \geq 0$ (de exemplu). Avem: $z \leq y + z \leq x + z \leq x + y + z \overset{(a)}{\Rightarrow}$

$f(y+z) + f(x+z) \overset{(a)}{\leq} f(z) + f(x+y+z) \Rightarrow f(x+y) + f(y+z) + f(x+z) \leq$

$f(x+y) + f(z) + f(x+y+z) \overset{(P1)}{\leq} f(x) + f(y) + f(z) + f(x+y+z)$;

II. Aplicam ipoteza inductiva $P(n)$ - adevarat, pentru numerele $x_1, ..., x_{n-1}, x_n + x_{n+1}$

III. $\sum_{1 \leq i \leq n-1} f(x_i + x_n + x_{n+1}) \overset{P(3)A}{\geq}$

$\sum_{1 \leq i \leq n-1} [f(x_i + x_n) + f(x_i + x_{n+1}) + f(x_n + x_{n+1}) - f(x_i) - f(x_n) - f(x_{n+1})]$;

IV. Din **II** si **III** rezulta:

$f(x_1 + ... + x_n + x_{n+1}) + (n-2)[f(x_1) + ... + f(x_{n-1}) + f(x_n + x_{n+1})] \geq$

$\sum_{1 \leq i \leq n-1} [f(x_i + x_n) + f(x_i + x_{n+1}) - f(x_i)] + (n-1)[f(x_n + x_{n+1}) - f(x_n) - f(x_{n+1})] \Leftrightarrow$

$f(x_1 + ... + x_n + x_{n+1}) + (n-1)[f(x_1) + ... + f(x_{n-1}) + f(x_n) + f(x_{n+1})] \geq$

$\sum_{1 \leq i < j \leq n+1} f(x_i + x_j) \Rightarrow P(n+1)A$

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3.

a. In planul complex de origine O, consideram punctele A si B, de afixe nenule a si respectiv b. Aratati ca $S(OAB)=\frac{1}{4}\left|\bar{a}\cdot b-a\cdot\bar{b}\right|$, unde $S(OAB)$ reprezinta aria triunghiului OAB.

b. Fie ABC un triunghi echilateral inscris intr-un cerc C de centru O. Pentru un punct P interior cercului C, notam cu $S(P)$ aria triunghiului avand laturile egale cu distantele de la punctul P la varfurile triunghiului ABC. Fie P_1 si P_2 doua puncte distincte interioare cercului C. Aratati ca $S(P_1)=S(P_2)$ daca si numai daca $OP_1=OP_2$.

Solutie.

$$\left.\begin{aligned} &S(OAB)=\frac{1}{2}OA\cdot OB\cdot\sin(\angle AOB)\\ &a)\sin(\arg(z))=\frac{1}{2\cdot i}\left(\frac{z-\bar{z}}{|z|}\right)\\ &\sin(\angle AOB)=\left|\sin\left(\arg\frac{a}{b}\right)\right| \end{aligned}\right\}\Rightarrow S(OAB)=\frac{1}{4}\left|\bar{a}b-a\bar{b}\right|.$$

$b)$Putem presupune $A(1), B(\varepsilon), C(\varepsilon^2), \varepsilon\in U_3\setminus\{1\}$.

$Fie\ P(p), D(p-1), E(\varepsilon(p-\varepsilon)), F(\varepsilon^2(p-\varepsilon^2))$. Folosim urmatoarea schema de rezolvare:

$\mathbf{I.}\ OD=PA, OE=PB, OF=PC;$

$\mathbf{II.}\ S(P)=S(ODE)=S(OEF)=S(OFD)=\frac{\sqrt{3}}{4}\left(1-|p|^2\right);$

$\mathbf{III.}\ S(P_1)=S(P_2)\Leftrightarrow OP_1=OP_2.$

Intr-adevar:

$\mathbf{I.}\ OD=|p-1|=|z_P-z_A|=PA, OE=|\varepsilon(p-\varepsilon)|=|p-\varepsilon|=|z_P-z_B|=PB;$

$OF=\left|\varepsilon^2\left(p-\varepsilon^2\right)\right|=\left|p-\varepsilon^2\right|=\left|z_p-z_C\right|=PC.$

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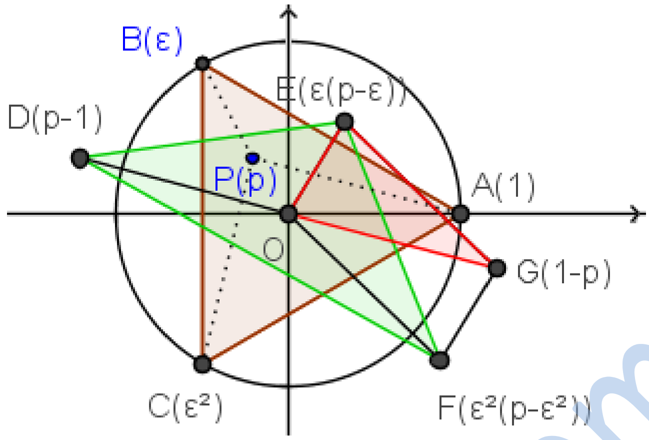

II. Fie punctul $G(1-p)$. Avem:

$z_D + z_E + z_F = 0 \Rightarrow O =$ centrul de greutate al triunghiului $DEF \Rightarrow$

$$S(ODE) = S(OEF) = S(OFD) \overset{D(p-1),G(1-p)}{=} S(OEG) \quad (1);$$

$$\left.\begin{array}{l} OE \overset{\mathbf{I}}{=} PB, \\ OG = OD \overset{\mathbf{I}}{=} PA, \\ EG \overset{OEGF\ paralelogram}{=} OF \overset{\mathbf{I}}{=} PC \end{array}\right\} \Rightarrow S(OED) = S(P) \quad (2)$$

$$S(OED) \overset{(a)}{=} \frac{1}{4}\left| \overline{\varepsilon(p-\varepsilon)}\cdot(p-1) - \varepsilon(p-\varepsilon)\overline{(p-1)} \right| =$$

$$\frac{1}{4}\left[\varepsilon^2\left(\bar{p}-\varepsilon^2\right)(p-1) - \varepsilon(p-\varepsilon)(\bar{p}-1)\right]$$

$$\frac{1}{4}\left|(2\varepsilon+1)\left(1-|p|^2\right)\right| = \frac{\sqrt{3}}{4}\left(1-|p|^2\right) \overset{(1),(2)}{\Rightarrow}$$

$$S(P) = S(ODE) = S(OEF) = S(OFD) = \frac{\sqrt{3}}{4}\left(1-|p|^2\right);$$

III. Daca p_1 si p_2 sunt afixele punctelor P_1 respectiv P_2 avem:

$$S(P_1) = S(P_2) \overset{\mathbf{II}}{\Leftrightarrow} \frac{\sqrt{3}}{4}\left(1-|p_1|^2\right) = \frac{\sqrt{3}}{4}\left(1-|p_2|^2\right) \Leftrightarrow |p_1| = |p_2| \Leftrightarrow OP_1 = OP_2.$$

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4.

Oamenii unui trib stravechi foloseau o limba in care cuvintele erau formate doar cu literele A *si* B. Cercetatorii au descoperit ca pentru oricare doua cuvinte de lungimi egale, exista cel putin trei pozitii corespondente in care literele sunt diferite. De exemplu, cuvintele $ABBAA$ *si* $AAAAB$ difera in pozitiile $2,3$ *si* 5, adica in trei pozitii.

Fie $n\in\mathbb{N}, n\geq 3$. Demonstrati ca in aceasta limba nu pot exista mai mult de $\left[\frac{2^n}{n+1}\right]$ cuvinte de lungime n ($[a]$ este partea intreaga a numarului real a)

Solutie.

Facem urmatoarele notatii:

C =multimea tuturor cuvintelor de lungime n care *s-ar putea forma* in limba respectiva (cuprinde eventual si cuvinte care nu fac parte din limba);

D= multimea tuturor cuvintelor de lungime n din limba respectiva $(D\subset C)$;

$d(x,y)$ =numarul de pozitii in care difera doua cuvinte $x,y\in C$;

$C_a=\{x\in C \mid d(x,a)\leq 1\}, a\in C.$

Folosim urmatoarea schema de rezolvare:

I. $card(D)\leq card(C)=2^n$; **II.** $a,b\in D$ *si* $a\neq b\Rightarrow C_a\cap C_b=\varnothing$;

III. $card(C_a)=n+1, \forall a\in C$; **IV.** $card(D)\leq\left[\frac{2^n}{n+1}\right]$.

Intr-adevar:

I. $D\subset C\Rightarrow card(D)\leq card(C)=2^n$;

II. Fie prin **R.A.** $a,b\in D$ *si* $x\in C_a\cap C_b\Rightarrow\begin{cases}d(x,a)\leq 1\\ d(b,x)\leq 1\end{cases}\Rightarrow$

$d(b,a)\leq 2\overset{a,b\in D}{\Rightarrow}$ *Contradictie*;

III. C_a contine pe a si inca n cuvinte obtinute prin modificarea lui a in cate o pozitie din cele n;

$$\bigcup_{a\in D}C_a\subset C\Rightarrow card\left(\bigcup_{a\in D}C_a\right)\overset{\mathbf{I}}{\leq}2^n\overset{\mathbf{II}}{\Leftrightarrow}\sum_{a\in D}card(C_a)\leq 2^n\overset{\mathbf{III}}{\Leftrightarrow}card(D)\cdot(n+1)\leq 2^n\Leftrightarrow$$

$$card(D)\leq\frac{2^n}{n+1}\overset{card(D)\in\mathbb{N}}{\Leftrightarrow}card(D)\leq\left[\frac{2^n}{n+1}\right]$$