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ETAPA FINALA - 2015

1.

Sa se gaseasca tripletele (a,b,c) de numere complexe nenule avand acelasi modul, care verifica egalitatea $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1=0$.

Solutie. Avem:

$$\begin{cases} \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1=0 \\ \overline{\frac{a}{b}}+\overline{\frac{b}{c}}+\overline{\frac{c}{a}}+1=0 \end{cases} \Leftrightarrow \begin{cases} \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1=0 \\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}+1=0 \end{cases} \Leftrightarrow$$

$$\begin{cases} a^2c+b^2a+c^2b+abc=0 \\ b^2c+c^2a+a^2b+abc=0 \end{cases} \overset{adunare}{\Leftrightarrow}$$

$$\begin{cases} a^2c+b^2a+c^2b+abc=0 \\ (a+b)(b+c)(c+a)=0 \end{cases} \quad (1).$$

Daca $b=-a \overset{(1)}{\Rightarrow} a^2c+a^3-c^2a-a^2c=0 \Leftrightarrow a(a-c)(a+c)=0$

$\Leftrightarrow (a-c)(a+c)=0 \Rightarrow (a,b,c)\in\{(a,-a,a),(a,-a,-a)\}$.

In final se obtin tripletele $(a,a,-a)$ cu $a\in\mathbb{C}^*$

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2.

Se considera un numar natural $k \geq 1$, numerele prime distincte $p_1, p_2, ..., p_k$ si $n = p_1 p_2 .... p_k$. Pentru o functie $f : \{1,2,....,n\} \to \{1,2,....,n\}$ notam $p(f) = f(1) f(2) .... f(n)$.

a. Sa se determine numarul functiilor f cu proprietatea ca $p(f)$ divide n.

b. Pentru $n = 6$, sa se afle numarul functiilor f pentru care $p(f)$ divide 36.

Solutie.

a)Daca $p(f) \mid n$, atunci $p(f) = p_1^{a_1} p_2^{a_2} .....p_k^{a_k}$, unde $a_k \in \{0,1\}$. Deducem un factor prim p_i fie nu divide $p(f)$ $(cazul\, a_i = 0)$, fie apare in descompunerea in factori primi a exact unuia dintre numerele $f(1), f(2), ..., f(n)$ $(cazul\, a_i = 1)$. Deci pentru fiecare din numerele $p_1, p_2, ..., p_k$ avem $n+1$ posibilitati de alegere, de unde rezulta ca numarul de functii cu proprietatea ceruta este $(n+1)^k$.

b)Din $p(f) \mid 36$ rezulta ca $p(f) = f(1) f(2) f(3) f(4) f(5) f(6) = 2^a 3^b$, $a, b \in \{0,1,2\}$ (1). Distingem astfel urmatoarele cazuri:

$Cazul\,1 : b = 0$:

Din (1) rezulta ca $p(f) = 2^a$, $a \in \{0,1,2\}$ si avem subcazurile:

$$\left.\begin{array}{l} Cazul\,1.1 : a = 0 \Rightarrow C_6^0 \ functii; \\ Cazul\,1.2 : a = 1 \Rightarrow C_6^1 \ functii; \\ Cazul\,1.3 : a = 2 \Rightarrow C_6^2 + C_6^1 \ functii \end{array}\right\} \Rightarrow 28 \ functii$$

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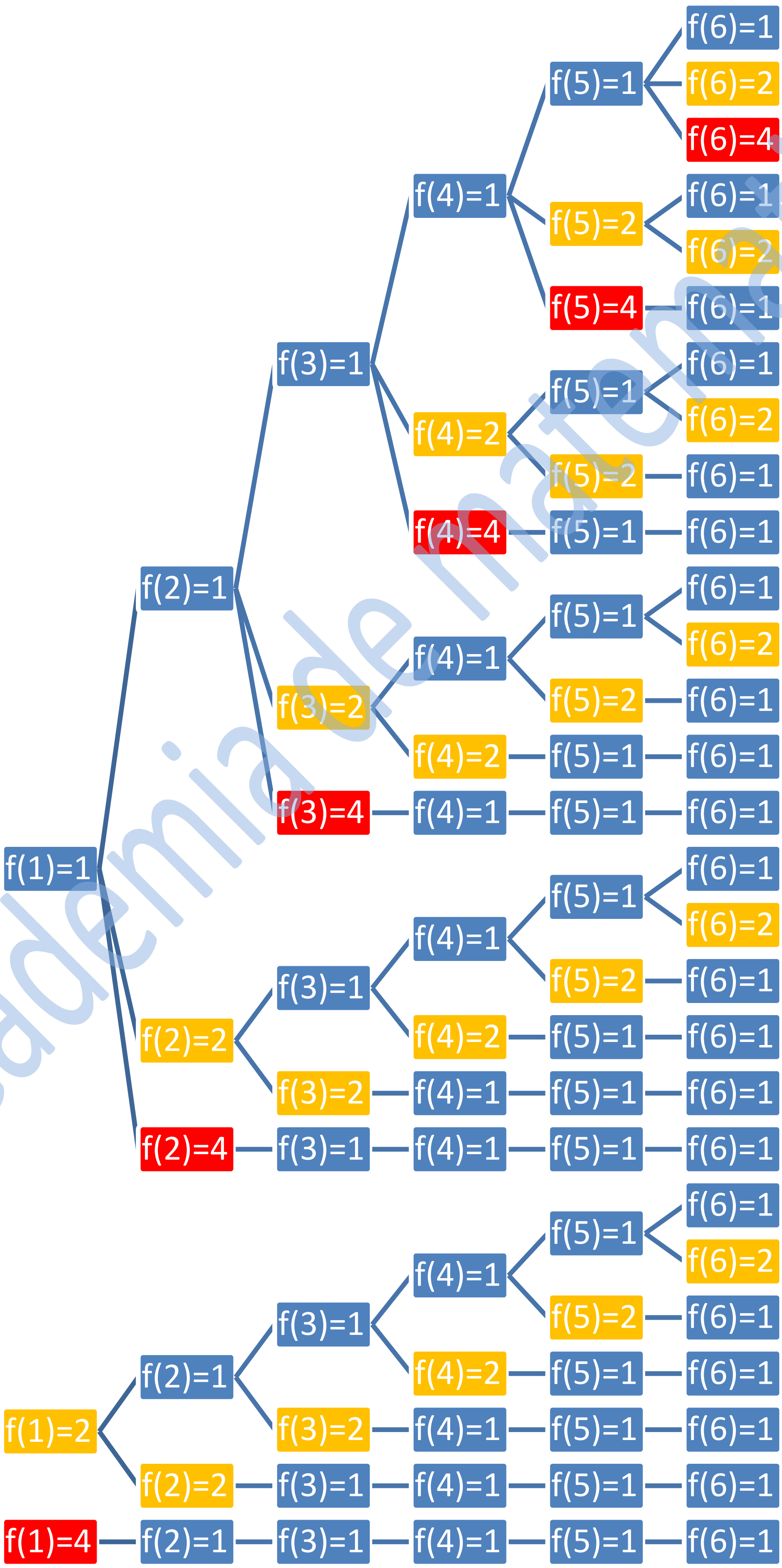

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$Cazul\,2: b=1:$

Din (1) rezulta ca $p(f)=f(1)f(2)f(3)f(4)f(5)f(6)=$

$2^a\cdot 3, a\in\{0,1,2\}$ si avem subcazurile:

$$\left.\begin{array}{l} Cazul\ \ 2.1: a=0 \Rightarrow C_6^1 C_6^0\ \ functii; \\ Cazul\ \ 2.2: a=1 \Rightarrow C_6^1 C_6^1\ \ functii; \\ Cazul\ \ 2.3: a=2 \Rightarrow C_6^1\left(C_6^2+C_5^1\right)\ functii \end{array}\right\} \Rightarrow 162\ functii$$

$Cazul\,3: b=2:$

Din (1) rezulta ca $p(f)=f(1)f(2)f(3)f(4)f(5)f(6)=2^a\cdot 3^2, a\in\{0,1,2\}\Rightarrow$

$$\underbrace{C_6^2}_{3\cdot 3}\left(\underbrace{C_6^0}_{2^0}+\underbrace{C_6^1}_{2}++\underbrace{C_6^2}_{2\cdot 2}+\underbrace{C_4^2}_{2^2}\right)=390\ functii$$

In concluzie se obtin $28+162+390=580$ functii cu proprietatea din enunt.

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3.

Sa se determine functiile $f:\mathbb{Q}\to\mathbb{Q}$ care verifica conditiile:

$$f\left(g(x)+g(y)\right)=f\left(g(x)\right)+y$$
$$g\left(f(x)+f(y)\right)=g\left(f(x)\right)+y$$

pentru orice $x,y\in\mathbb{Q}$

Solutie. Folosim urmatoarea schema de rezolvare:

I. f,g *injective si* $f(0)=g(0)=0$; **II.** f,g *bijective si* $g=f^{-1}$;

III. f,g *aditive*; **IV.** $f(x)=ax,\ g(x)=\frac{x}{a},\ a\in\mathbb{Q}^*$.

Intr-adevar:

I. *Fie* $y_1,y_2\in\mathbb{Q}$ *a.i.* $g\left(y_1\right)=g\left(y_2\right)\overset{enunt}{\Rightarrow} f\left(g(x)\right)+y_1=f\left(g(x)\right)+y_2\Rightarrow$

$y_1=y_2\Rightarrow g$ *injectiva* $\overset{anlog}{\Rightarrow} f$ *injectiva* Apoi $f\left(g(x)+g(0)\right)=f\left(g(x)\right)\overset{g\ injectiva}{\Rightarrow}$

$g(0)=0\overset{analog}{\Rightarrow} f(0)=0.$

II. $f\left(g(0)+g(y)\right)=f\left(g(0)\right)+y\overset{\mathbf{I}}{\Leftrightarrow} f\left(g(y)\right)=y$ *si analog*
$g\left(f(y)\right)=y,\forall y\in\mathbb{Q}.$

III. $f\left(g(x)+g(y)\right)=f\left(g(x)\right)+y\overset{\mathbf{II}}{\Rightarrow} f\left(g(x)+g(y)\right)=x+y\Rightarrow$

$g\left[f\left(g(x)+g(y)\right)\right]=g(x+y)\overset{\mathbf{II}}{\Rightarrow} g(x+y)=g(x)+g(y)$ *si analog*
$f(x+y)=f(x)+f(y),\forall x,y\in\mathbb{Q};$

IV. Din **III** si din $f,g:\mathbb{Q}\to\mathbb{Q}\Rightarrow f(x)=f(1)x=ax\overset{g=f^{-1}}{\Rightarrow} g(x)=\frac{x}{a}.$

Se arata imediat ca aceste functii verifica conditiile enuntului.

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4.
Fie A o multime finite de numere reale. Consideram multimile:

$$S=\{x+y\,|\,x,y\in A\},\quad D=\{x-y\,|\,x,y\in A\}\ .$$

Sa se arate ca $card(A)\cdot card(D)\le[card(S)]^2$.

Solutie. Consideram functia $f:A\times D\to S\times S$ definita prin $f(a,d)=(a+x_d,a+y_d)$, unde $x_d,y_d\in A$ astfel incat $x_d-y_d=d$ si x_d este *maxim posibil.* In acest fel $x_d,y_d\in A$ sunt unice si functia f este correct definita. Folosim urmatoarea schema de rezolvare:

$$\textbf{I.}\ f\ injectiva;\quad \textbf{II.}\ card(A)\cdot card(D)\le[card(S)]^2;$$

Intr-adevar:

$$\textbf{I.}\ f[(a_1,d_1)]=f[(a_2,d_2)]\Leftrightarrow(a_1+x_{d_1},a_1+y_{d_1})=(a_2+x_{d_2},a_2+y_{d_2})\Leftrightarrow$$

$$\begin{cases}a_1+x_{d_1}=a_2+x_{d_2}\\ a_1+y_{d_1}=a_2+y_{d_2}\end{cases}\Leftrightarrow\begin{cases}a_1+x_{d_1}=a_2+x_{d_2}\\ x_{d_1}-y_{d_1}=x_{d_2}-y_{d_2}\end{cases}\Leftrightarrow\begin{cases}a_1+x_{d_1}=a_2+x_{d_2}\\ d_1=d_2\end{cases}\Rightarrow$$

$$\begin{cases}a_1=a_2\\ d_1=d_2\end{cases}\Rightarrow f\ injectiva\quad \textbf{II.}\ \text{Din}\ \textbf{I}.$$

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