OLIMPIADA NATIONALA DE MATEMATICA
ETAPA FINALA - 2014

1.

Fie n un numar natural nenul. Pentru fiecare numar natural nenul k notam cu $a(k,n)$ numarul divizorilor naturali d ai lui k astfel incat $k \le d^2 \le n^2$. Sa se calculeze $\sum_{k=1}^{n^2} a(k,n)$.

Solutie. Consideram un tabel cu n linii si n^2 coloane pe care il completam astfel: in casuta de pe linia d si colana k punem numarul 1 daca $d \mid k, k \le d^2 \le n^2$ si numarul 0 in caz contrar.

	(1)			(d)		(2d)		(d^2)		(n^2)
(1)	1	0	---------	0	---------	0	---------	0	---------	0
	-----	-----	-----	-----	---------	-----		-----		-----
(d)	0	0	---------	1	---------	1	---------	1	---------	0
	-----	-----	---------	-----	---------	-----	---------	----	---------	-----
(n)	-----	-----	---------	-----	---------	-----	---------		---------	-----

Adunand numerele din tabel mai intai pe coloane si apoi pe linii se obtine suma $\sum_{k=1}^{n^2} a(k,n)$. Adunand numerele din tabel mai intai pe linii si apoi pe coloane se obtine suma $1+2+.....+d+.....+n=\frac{n(n+1)}{2}$, deci $\sum_{k=1}^{n^2} a(k,n)=\frac{n(n+1)}{2}$.

OLIMPIADA NATIONALA DE MATEMATICA
ETAPA FINALA - 2014

2.

Fie n un numar natural nenul si $A=\{1,2,....,n\}$. Sa se determine numarul functiilor crescatoare $f:A\to A$ cu proprietatea ca $|f(x)-f(y)|\le|x-y|$ pentru orice $x,y\in A$.

Solutie. Notam cu $\mathscr{F}$ multimea functiilor cu proprietatea din enunt si cu $\mathscr{F}_l=\{f\in\mathscr{F}\mid f(1)=l\}$ si folosim urmatoarea schema de rezolvare:

I. $f\in\mathscr{F}\Rightarrow f(k+1)-f(k)\in\{0,1\},\forall k\in(1,2,...,n-1\}$;

II. $card(\mathscr{F}_l)=C_{n-1}^0+C_{n-1}^1+....+C_{n-1}^{n-l}$; **III.** $card(\mathscr{F})=(n+1)\cdot 2^{n-2}$.

Intr-adevar:

I. Din $|f(k+1)-f(k)|\le|k+1-k|$ si din faptul ca f este crescatoare;

II. Daca $f(k+1)-f(k)=1$ vom spune ca functia f "*are un salt*" in *punctul* $k+1$.

O functie $f\in\mathscr{F}_l$ are *maxim* $n-l$ salturi si este perfect determinata de *numarul de salturi* si *punctele* in care functia are aceste salturi, de unde rezulta $card(\mathscr{F}_l)=C_{n-1}^0+C_{n-1}^1+....+C_{n-1}^{n-l}$.

III. $\left.\begin{array}{l}\mathscr{F}=\mathscr{F}_1\cup.....\cup\mathscr{F}_n\\ \mathscr{F}_1,.....,\mathscr{F}_n \text{ disjuncte } 2 \text{ cate } 2\end{array}\right\}\Rightarrow card(\mathscr{F})=\sum_{l=1}^{n}card(\mathscr{F}_l)=$

$$\sum_{l=1}^{n}\left(C_{n-1}^0+C_{n-1}^1+....+C_{n-1}^{n-l}\right)=\sum_{k=0}^{n-1}(n-k)C_{n-1}^k=n\sum_{k=0}^{n-1}C_{n-1}^k-\sum_{k=0}^{n-1}kC_{n-1}^k=$$

$$n\cdot 2^{n-1}-(n-1)\sum_{k=0}^{n-2}C_{n-2}^k=n\cdot 2^{n-1}-2^{n-2}(n-1)=(n+1)\cdot 2^{n-2}.$$

OLIMPIADA NATIONALA DE MATEMATICA ETAPA FINALA - 2014

3.

Consideram functia $f:\mathbb{N}^* \to \mathbb{N}^*$ cu proprietatile:

$a)\ f(1)=1,$

$b)\ f(p)=1+f(p-1)$ pentru orice numar prim p,

$c)\ f(p_1p_2....p_n)=f(p_1)+f(p_2)+.....+f(p_n)$ pentru orice numere prime $p_1,\ p_2,....,p_n$ nu neaparat distincte.

Sa se arate ca $2^{f(n)}\le n^3\le 3^{f(n)}$ pentru orice numar natural $n, n\ge 2$.

Solutie.

Inegalitatile $2^{f(n)}\le n^3\le 3^{f(n)}$ sunt echivalente cu dubla inegalitate $3\log_3 n\le f(n)\le 3\log_2 n$ pe care o demonstram prin *inductie*. Avem:

$f(2)\overset{(b)}{=}1+f(1)=2$ si $3\log_3 2\le f(2)\le 3\log_2 2$

Fie apoi $3\log_3 n\le f(n)\le 3\log_2 n$ (1) si demonstram ca $3\log_3(n+1)\le f(n+1)\le 3\log_2(n+1).$

$Cazul\,1: n+1=p_1p_2.....p_k:$

$f(n+1)\overset{(c)}{=}f(p_1)+f(p_2)+.....+f(p_k)\overset{(1)}{\ge}$
$3\log_3 p_1+3\log_3 p_1+.....+3\log_3 p_k=3\log_3 p_1p_2....p_k=3\log_3(n+1).$

Apoi $f(n+1)\overset{(c)}{=}f(p_1)+f(p_2)+.....+f(p_k)\overset{(1)}{\le}$
$3\log_2 p_1+3\log_2 p_1+.....+3\log_2 p_k=3\log_2 p_1p_2....p_k=3\log_2(n+1).$

$Cazul\,2: n+1\ prim:$

Avem $f(n+1)=1+f(n)=1+f\left(2\cdot\frac{n}{2}\right)\overset{n\ par\ (c)}{=}1+f(2)+f\left(\frac{n}{2}\right)=3+f\left(\frac{n}{2}\right)\overset{(1)}{\ge}$

$3+3\log_3\frac{n}{2}=3\log_3\frac{3n}{2}\ge 3\log_3(n+1).$

Apoi $f(n+1)=3+f\left(\frac{n}{2}\right)\overset{(1)}{\le}3+3\log_2\frac{n}{2}=3\log_2 n\le 3\log_2(n+1).$

OLIMPIADA NATIONALA DE MATEMATICA
ETAPA FINALA - 2014

4.

Fie n un numar intreg $n \geq 2$ si numerele complexe $a_1, a_2, ..., a_n$ cu $a_n \neq 0$. Sa se arate ca urmatoarele afirmatii sunt echivalente:

$a) \left| a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 \right| \leq \left| a_n + a_0 \right|$, pentru orice numar complex z de modul egal cu 1;

$b) a_1 = a_2 = ... = a_{n-1} = 0 \;\; si \;\; \frac{a_0}{a_{n-1}} \in [0, \infty)$.

Solutie.

$b \Rightarrow a:$

$$\left| a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 \right| = \left| a_n z^n + a_0 \right| \overset{|z|=1}{\leq} |a_n| + |a_0| \overset{\frac{a_0}{a_n} \in [0,\infty)}{=} |a_n + a_0|;$$

$a \Rightarrow b$: Fie $w = a_n + a_0 \;\; si \;\; g(z) = a_{n-1} z^{n-1} + ... + a_1 z$. Folosim urmatoarea schema de rezolvare:

I. $|w + g(\varepsilon)| \leq |w|, \forall \varepsilon \in U_n$; **II.** $g(\varepsilon) = 0, \forall \varepsilon \in U_n$;

III. $a_1 = a_2 = ... = a_{n-1} = 0$; **IV.** $\frac{a_0}{a_n} \in [0, \infty)$.

Intr-adevar:

I. Din enunt; **II.** Din **I** rezulta $|g(\varepsilon)|^2 + w \cdot \overline{g(\varepsilon)} + \overline{w} \cdot g(\varepsilon) \leq 0$ si prin sumare:

$$\sum_{\varepsilon \in U_n} |g(\varepsilon)|^2 + w \sum_{\varepsilon \in U_n} \overline{g(\varepsilon)} + \overline{w} \sum_{\varepsilon \in U_n} g(\varepsilon) \leq 0 \overset{\substack{\sum_{\varepsilon \in U_n} \varepsilon^k = 0 \\ k \in \{1,2,...,n-1\}}}{\Leftrightarrow} \sum_{\varepsilon \in U_n} |g(\varepsilon)|^2 \leq 0 \Rightarrow$$

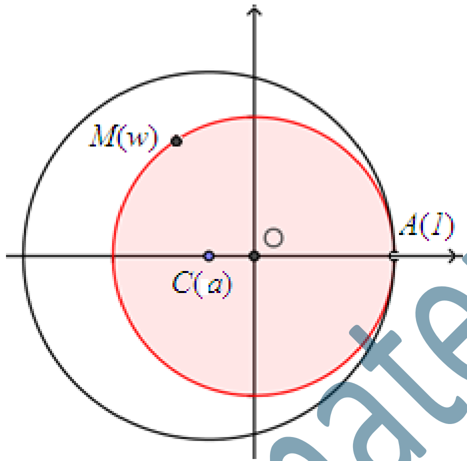
$$g(\varepsilon) = 0, \forall \varepsilon \in U_n.$$

III. *Fie* $k \in \{1, 2, ..., n-1\}$. Avem: $g(\varepsilon) \overset{\mathbf{II}}{=} 0 \Leftrightarrow \frac{g(\varepsilon)}{\varepsilon^k} = 0 \Rightarrow$

$$\sum_{\varepsilon \in U_n} \frac{g(\varepsilon)}{\varepsilon^k} = 0 \overset{\substack{\sum_{\varepsilon \in U_n} \varepsilon^i = 0 \\ i \in \{\pm 1, \pm 2, ..., \pm(n-1)\}}}{\Rightarrow} n a_k = 0 \Rightarrow a_k = 0;$$

OLIMPIADA NATIONALA DE MATEMATICA
ETAPA FINALA - 2014

IV. Fie $a=-\frac{a_0}{a_n}$, $w=z^n$, $C(a)$, $A(1)$, $M(w)$. Avem:



$$\left|a_n z^n + a_0\right| \overset{III}{\le} |a_n+a_0|, \forall z\in\mathbb{C}, |z|=1 \Leftrightarrow$$

$$\left|z^n+\frac{a_0}{a_n}\right| \le \left|1+\frac{a_0}{a_n}\right|, \forall z\in\mathbb{C}, |z|=1 \Leftrightarrow |w-a|\le|1-a|, \forall w\in\mathbb{C}, |w|=1\Leftrightarrow$$

$$CM\le CA, \forall M\in\mathcal{C}(O,1)\Rightarrow \mathcal{C}(O,1)\subset\mathcal{C}(C,CA) \overset{A\in\mathcal{C}(O,1)\cap\mathcal{C}(C,CA)}{\Rightarrow}$$

cercul $\in \mathcal{C}(O,1)$ este tangent interior cercului $\mathcal{C}(C,CA)$ in $A \Rightarrow CA\ge 1\Rightarrow$

$$a\le 0 \Leftrightarrow \frac{a_0}{a_n}\in[0,\infty).$$