OLIMPIADA NATIONALA DE MATEMATICA
ETAPA FINALA - 2013

1.

Sa se rezolve ecuatia $2^{\sin^4 x-\cos^2 x}-2^{\cos^4 x-\sin^2 x}=2\cos 2x$.

Solutie.

$$2^{\sin^4 x-\cos^2 x}-2^{\cos^4 x-\sin^2 x}=2\cos 2x \Leftrightarrow$$

$$2^{\sin^4 x-\cos^2 x}-2^{\cos^4 x-\sin^2 x}=\cos^4 x-\sin^2 x-\left(\sin^4 x-\cos^2 x\right)\Leftrightarrow$$

$$2^{\sin^4 x-\cos^2 x}+\sin^4 x-\cos^2 x=2^{\cos^4 x-\sin^2 x}+\cos^4 x-\sin^2 x \quad (1).$$

Cum functia $f:R\to R, f(t)=2^t+t$ este strict crescatore, deci injectiva, rezulta tinand cont de (1):

$$\sin^4 x-\cos^2 x=\cos^4 x-\sin^2 x \Leftrightarrow 2\cos 2x=0 \Leftrightarrow x\in\left\{\frac{(2k+1)\pi}{4}\,\middle|\,k\in\mathbb{Z}\right\}$$

2.

Sa se determine toate functiile injective $f:\mathbb{Z}\to\mathbb{Z}$ care satisfac relatia $|f(x)-f(y)|\le|x-y|$ pentru orice $x,y\in\mathbb{Z}$.

Solutie.

Folosim urmatoarea schema de rezolvare:

$\mathbf{I.}\ f(x+1)-f(x)\in\{-1,1\},\forall x\in\mathbb{Z};\quad \mathbf{II.}\ f(x)=x+f(0)\ sau\ f(x)=-x+f(0);$

Intr-adevar:

$$\mathbf{I.}\,|f(x+1)-f(x)|\le 1\Rightarrow f(x+1)-f(x)\in\{-1,0,1\}\overset{f\ injectiva}{\Rightarrow}$$

$$f(x+1)-f(x)\in\{-1,1\};$$

$$\mathbf{II.}\ Cazul\ 1:\ f(1)=1+f(0)\overset{f(2)\ne f(0)}{\Rightarrow} f(2)=1+f(1)=2+f(0)\overset{inductiv}{\Rightarrow}$$

$$\begin{cases} f(n)=n+f(0)\\ f(-n)=-n+f(0)\end{cases},\ \forall n\in\mathbb{N};$$

$$Cazul\ 2:\ f(1)=-1+f(0)\overset{analog}{\Rightarrow} f(x)=-x+f(0),\forall x\in\mathbb{Z}.$$

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3.

Se considera numerele complexe distincte a,b,c,d. Sa se demonstreze ca urmatoarele afirmatii sunt echivalente:

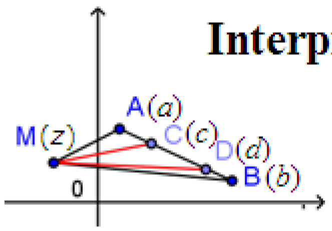
i)Pentru orice $z\in\mathbb{C}$ are loc inegalitatea $|z-a|+|z-b|\geq|z-c|+|z-d|$;

ii)Exista $t\in(0,1)$ astfel incat $c=ta+(1-t)b$ si $d=(1-t)a+tb$.

Interpretare geometrica.

Solutie. Pentru interpretarile geometrice vom considera in planul complex punctele $A(a),B(b),C(c),D(d),M(z)$.

$ii)\Rightarrow i):|z-c|+|z-d|=|z-ta-(1-t)b|+|z-(1-t)a-tb|\leq$
$[t|z-a|+(1-t)|z-b|]+[(1-t)|z-a|+t|z-b|]=|z-a|+|z-b|$.



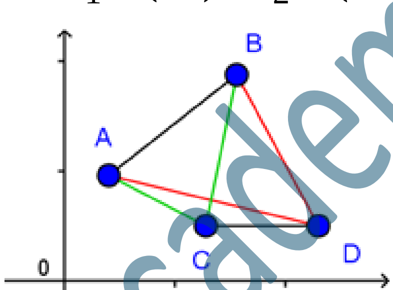
Interpretare geometrica: $C,D\in(AB)$ si $AC=BD\Rightarrow$

$MA+MB\geq MC+MD$

$i)\Rightarrow ii)$: Folosim urmatoarea schema de rezolvare:

I. $|a-b|=|a-c|+|b-c|=|a-d|+|b-d|=|a-c|+|a-d|=|b-c|+|b-d|$;

II. $\exists t_1\in(0,1),\exists t_2\in(0,1)$ a.i. $c=t_1a+(1-t_1)b$ si $d=t_2a+(1-t_2)b$; **III.** $t_1+t_2=1$;



Intr-adevar:

I. $\left.\begin{array}{l} z=a\Rightarrow|a-b|\geq|a-c|+|a-d| \\ z=b\Rightarrow|a-b|\geq|b-c|+|b-d| \end{array}\right\}\Rightarrow$

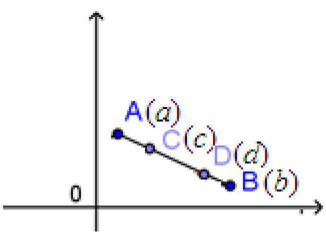
$|a-c|+|a-d|+|b-c|+|b-d|\leq 2|a-b|\quad(1)$;

$2|a-b|=|a-c+c-b|+|b-d+d-a|\leq|a-c|+|b-c|+|a-d|+|b-d|(2)$;

$Din\ (1)\ si\ (2)\Rightarrow 2|a-b|\leq|a-c|+|b-c|+|a-d|+|b-d|\leq 2|a-b|\Rightarrow$

$2|a-b|=|a-c|+|b-c|+|a-d|+|b-d|\overset{(1),(2)}{\Rightarrow}$

$|a-b|=|a-c|+|b-c|=|a-d|+|b-d|=|a-c|+|a-d|=|b-c|+|b-d|$;



II. Din **I** cu **interpretarea geometrica** $C,D\in(AB)$;

III. $|a-c|+|b-c|\overset{\mathbf{I}}{=}|b-c|+|b-d|\Rightarrow|a-c|=|b-d|\overset{\mathbf{II}}{\Rightarrow}$

$(1-t_1)|a-b|=t_2|a-b|\Rightarrow t_1+t_2=1$

cu **interpretarea geometrica** $C,D\in(AB)$ si $AC=DB$.

OLIMPIADA NATIONALA DE MATEMATICA
ETAPA FINALA - 2013

4.

a. Sa se arate ca $\frac{1}{2}+\frac{1}{3}+....+\frac{1}{2^m}<m$, pentru orice $m\in\mathbb{N}^*$.

b. Fie $p_1, p_2,..., p_n$ numere prime distincte mai mici decat 2^{100}. Sa se arate ca

$$\frac{1}{p_1}+.....+\frac{1}{p_n}<10.$$

Solutie.

$$a)\frac{1}{2}+\frac{1}{3}+....+\frac{1}{2^m}=\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+....+\left(\frac{1}{2^{m-1}+1}+.....+\frac{1}{2^{m-1}+2^{m-1}}\right)<$$

$$\frac{1}{2}+2\cdot\frac{1}{2}+2^2\cdot\frac{1}{2^2}.....+2^{m-1}\cdot\frac{1}{2^{m-1}}=\frac{1}{2}+(m-1)=m-\frac{1}{2}<m.$$

$b)$Avem:

$$\left(\frac{1}{p_1}+.....+\frac{1}{p_n}\right)^4=\sum_{\substack{k_1,k_2,k_3,k_4\in\mathbb{N}\\ k_1+k_2+k_3+k_4=4}}\frac{4!}{k_1!k_2!k_3!k_4!}\cdot\frac{1}{p_1^{k_1}p_2^{k_2}p_3^{k_3}p_4^{k_4}}\le$$

$$4!\cdot\sum_{\substack{k_1,k_2,k_3,k_4\in\mathbb{N}\\ k_1+k_2+k_3+k_4=4}}\frac{1}{p_1^{k_1}p_2^{k_2}p_3^{k_3}p_4^{k_4}}=4!\cdot\sum_{1\le i\le j\le k\le l\le n}\frac{1}{p_ip_jp_kp_l}\quad(1).$$

Cum numerele $p_ip_jp_kp_l, 1\le i\le j\le k\le l\le n$ sunt distincte doua cate doua si mai mici decat 2^{400} tinand cont de (1) obtinem:

$$\left(\frac{1}{p_1}+.....+\frac{1}{p_n}\right)^4\le 4!\cdot\left(\frac{1}{2}+\frac{1}{3}+....+\frac{1}{2^{400}}\right)\overset{(a)}{<}4!\cdot400=9600<10^4,$$ de unde rezulta

$$\frac{1}{p_1}+.....+\frac{1}{p_n}<10.$$