OLIMPIADA NATIONALA DE MATEMATICA
ETAPA FINALA - 2017

1.

Spunem ca functia $f:\mathbb{R}\to\mathbb{R}$ are proprietatea (P) daca oricare ar fi un sir de numere reale $(x_n)_{n\geq 1}$ astfel incat sirul $(f(x_n))_{n\geq 1}$ este convergent, rezulta ca sirul $(x_n)_{n\geq 1}$ este convergent. Demonstrati ca o functie *surjectiva* cu proprietatea (P) este *continua*.

Solutie.

Folosim urrmatoarea schema de rezolvare:

$$\textbf{I.}\ \exists f^{-1}=g;\quad \textbf{II.}\ (y_n)\ convergent \Rightarrow (g(y_n))\ convergent;$$

$$\textbf{III.}\ g=f^{-1}\ \text{continua pe}\ \mathbb{R};\quad \textbf{IV.}\ f\ \text{continua pe}\ \mathbb{R};$$

Intr-adevar:

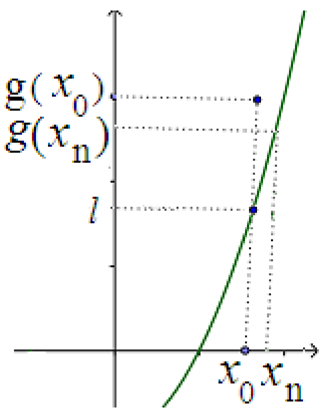
$$\textbf{I.}\ Fie\ a,b\in R\ a.i.\ f(a)=f(b)\ si\ x_n=\begin{cases}a, n=par\\ b, n=impar\end{cases}\Rightarrow (f(x_n))\ convergent \overset{(P)}{\Rightarrow}$$

$$(x_n)\ convergent \Rightarrow a=b\Rightarrow f\ injectiva$$

$$\textbf{II.}\ x_n=g(y_n)\Rightarrow y_n=f(x_n)\ \underset{ipoteza}{\overset{(y_n)\ convergent}{\Rightarrow}}\ (x_n)\ convergent$$

$$\textbf{III.}\ \text{Fie prin}\ \textbf{R.A.}\ x_0\in\mathbb{R}\ si\ (x_n)_{n\geq 1}\ astfel\ incat\ \lim_{n\to\infty}x_n=x_0\in\mathbb{R}\quad (1)$$

$$si\ \lim_{n\to\infty}g(x_n)\overset{\textbf{II}}{=}l\neq g(x_0)\quad (2).$$



$$\text{Fie}\ (y_n)_{n\geq 1},\ y_n=\begin{cases}x_n, n\ par\\ x_0, n\ impar\end{cases}\overset{(1)}{\Rightarrow}\lim_{n\to\infty}y_n=x_0\overset{\textbf{II}}{\Rightarrow}\lim_{n\to\infty}g(y_{2n})=\lim_{n\to\infty}g(y_{2n-1})\Leftrightarrow$$

$$\lim_{n\to\infty}g(x_{2n})=g(x_0)\overset{(2)}{\Rightarrow}l=g(x_0)-Contradicte\ cu\ (2).\ \textbf{III.}\ \text{Din}\ \textbf{I}\ \text{si}\ \textbf{II}.$$

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2.

Fie $A_1, A_2, ..., A_k \in \mathcal{M}_n(\mathbb{R})$ matrice simetrice. Demonstrati ca urmatoarele afirmatii sunt echivalente:

$1) \det\left(A_1^2 + A_2^2 + ... + A_k^2\right) = 0;$

$2) \det\left(A_1B_1 + A_2B_2 + ... + A_kB_k\right) = 0, \forall B_1, B_2, ..., B_k \in \mathcal{M}_n(\mathbb{R}).$

(O matrice X este simetrica daca coincide cu transpusa sa).

Solutie.

$"2 \Rightarrow 1": evident;$

$"1 \Rightarrow 2"$: Folosim urmatoarea schema de rezolvare:

I. $\exists X \in \mathcal{M}_{n,1}(\mathbb{R}), X \neq O_{n,1}$ *a.i.* $\left(A_1^2 + A_2^2 + ... + A_k^2\right)X = O_{n,1};$

II. $A_1X = A_2X = ... = A_kX = O_{n,1};$

III. $\left({}^tB_1{}^tA_1 + {}^tB_2{}^tA_2 + ... + {}^tB_k{}^tA_k\right)X = O_{n,1};$

IV. $\det\left(A_1B_1 + A_2B_2 + ... + A_nB_n\right) = 0.$

Intr-adevar:

I. Din $\det\left(A_1^2 + A_2^2 + ... + A_k^2\right) = 0;$

II. $\left(A_1^2 + A_2^2 + ... + A_k^2\right)X \overset{\mathbf{I}}{=} O_{n,1} \Rightarrow {}^tX\left(A_1^2 + A_2^2 + ... + A_k^2\right)X = 0 \Leftrightarrow$

${}^tXA_1A_1X + {}^tXA_2A_2X + ... + {}^tXA_kA_kX = 0 \overset{A_i \text{ simetrice}}{\Leftrightarrow}$

${}^t(A_1X)(A_1X) + {}^t(A_2X)(A_2X) + ... + {}^t(A_kX)(A_kX) = 0 \overset{{}^t(A_iX)(A_iX) = \text{suma patrate}}{\Leftrightarrow}$

${}^t(A_iX)(A_iX) = 0 \Leftrightarrow A_iX = O_{n,1};$

III. ${}^tA_iX \overset{\mathbf{II},\ A_i \text{, simetrica}}{=} O_{n,1} \Rightarrow {}^tB_i{}^tA_iX = O_{n,1} \Rightarrow$

$\left({}^tB_1{}^tA_1 + {}^tB_2{}^tA_2 + ... + {}^tB_k{}^tA_k\right)X = O_{n,1};$

IV. Din **III** si $X \neq O_{n,1} \Rightarrow \det\left({}^tB_1{}^tA_1 + {}^tB_2{}^tA_2 + ... + {}^tB_k{}^tA_k\right) = 0 \Rightarrow$

$\det\left(A_1B_1 + A_2B_2 + ... + A_kB_k\right) = 0.$

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3.

Fie $n \geq 2$ intreg si $A, B \in \mathcal{M}_n(\mathbb{C})$. Daca $(AB)^3 = O_n$, rezulta oare ca $(BA)^3 = O_n$?. Justificati raspunsul.

Solutie.

Fie $AB = C, BA = D, si\ p_C, p_D \in \mathbb{C}[X]$ polinoamele caracteristice ale matricilor C respectiv D. Folosim urmatoarea schema de rezolvare:

I. $\det(C) = \det(D) = 0$; **II.** $p_C = p_D$; **III.** $C^2 = O_n \Rightarrow D^3 = O_n$;

IV. $\left.\begin{matrix} C^2 \neq O_n \\ n = 2 \end{matrix}\right\} \Rightarrow D^3 = O_n$; **V.** $\left.\begin{matrix} C^2 \neq O_n \\ n = 3 \end{matrix}\right\} \Rightarrow D^3 = O_n$;

VI. $n \geq 4 \Rightarrow \exists A, B\ a.i.\ C^3 = O_n\ si\ D^3 \neq O_n$;

VII. Rezulta obligatoriu $(BA)^3 = O_n$ numai pentru $n \in \{2,3\}$;

Intr-adevar:

I. Din $C^3 = O_n$; **II.** Relatie cunoscuta;

III.: $C^2 = O_n :\Rightarrow D^3 = B(AB)^2 A = BC^2 A = O_n$;

IV. $p_C \overset{\mathbf{II}}{=} p_D \overset{\mathbf{I}}{=} X^2 - aX,\ a = tr(C) = tr(D) \overset{HC}{\Rightarrow} C^2 = aC \Rightarrow$

$$O_n = C^3 = aC^2 \overset{C^2 \neq O_n}{\Rightarrow} a = 0 \Rightarrow p_D = X^2 \Rightarrow D^2 = O_n \Rightarrow D^3 = O_n;$$

V. $p_C \overset{\mathbf{II}}{=} p_D \overset{\mathbf{I}}{=} X^3 - aX^2 + bX \quad (1) \Rightarrow C^3 - aC^2 + bC = O_n \overset{C^3 = O_n}{\Rightarrow}$

$$aC^2 = bC \quad (2) \Rightarrow aC^3 = bC^2 \overset{\substack{C^3 = O_n \\ C^2 \neq O_n}}{\Rightarrow} b = 0 \quad (3) \overset{(2)}{\Rightarrow} aC^2 = O_n \overset{C^2 \neq O_n}{\Rightarrow} a = 0 \overset{(1),(3)}{\Rightarrow}$$

$$p_D = p_C = X^3 \overset{ecuatia\ Hamilton-Cayley}{\Rightarrow} D^3 = O_n;$$

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VI. Pentru $n=4$ alegem $A=\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&0&1&0\end{pmatrix}$, $B=\begin{pmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&0&0\end{pmatrix}$ si obtinem

$(AB)^3=O_n\neq(BA)^3=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$, iar pentru $n\geq5$ alegem matricile bloc

$A'=\begin{pmatrix}O_{4,n\text{-}4} & \vert & A\\ - & - & -\\ & O_{n-4,n} & \end{pmatrix}$, $B'=\begin{pmatrix}O_{4,n\text{-}4} & \vert & B\\ - & - & -\\ & O_{n-4,n} & \end{pmatrix}$ (A, B cele anterioare si

obtinem $\left(A'B'\right)^3=O_n\neq\left(B'A'\right)^3$; **VII.** Din etapele anterioare

4.

Fie $f:[a,b]\to[a,b]$ o functie derivabila cu f' continua si strict pozitiva. Demonstrati ca exista $c\in(a,b)$ astfel incat

$$f(f(b))-f(f(a))=\left(f'(c)\right)^2(b-a).$$

Solutie.

Folosim urmatoarea schema de rezolvare:

I. $\exists c_1,c_2\in(a,b)$ *a.i.* $f(f(b))-f(f(a))=f'(c_1)f'(c_2)(b-a)$;

II. $\exists c$ intre c_1 *si* c_2 astfelincat $f'(c_1)f'(c_2)=\left(f'(c)\right)^2$;

III. $\exists c\in(a,b)$ astfelincat $f(f(b))-f(f(a))=\left(f'(c)\right)^2(b-a)$.

Intr-adevar:

I. Se aplica de doua ori succesiv teorema lui Lagrange;

II. $\sqrt{f'(c_1)f'(c_2)}$ este cuprins intre $f'(c_1)$ *si* $f'(c_2)$ $\overset{f'\ are\ PD}{\Rightarrow}$ $\exists c$ intre c_1 *si* c_2 astfel incat $f'(c_1)f'(c_2)=\left(f'(c)\right)^2$; **III.** Din **I** si **II**.