# OLIMPIADA NATIONALA DE MATEMATICA
# ETAPA FINALA - 2016

**1.**

Fie $A \in \mathcal{M}_2(\mathbb{R})$ o matrice care satisface urmatoarele conditii:
$\det(A^{2014} - I_2) = \det(A^{2014} + I_2)$ si $\det(A^{2016} - I_2) = \det(A^{2016} + I_2)$.

Demonstrati ca $\det(A^n - I_2) = \det(A^n + I_2)$, pentru orice numar natural nenul $n$.

**Solutie.**

Fie $Tr(A) = t, \det(A) = \delta$. Folosim urmatoarea schema de rezolvare:

$$\mathbf{I.}\ X \in \mathcal{M}_2(\mathbb{R})\ si\ \det(X - I_2) = \det(X + I_2) \Rightarrow Tr(X) = 0;$$

$$\mathbf{II.}\ Tr(A^{2014}) = Tr(A^{2016}) = 0; \quad \mathbf{III.}\ t = \delta = 0;$$

$$\mathbf{IV.}\ \det(A^n - I_2) = \det(A^n + I_2),\ \forall n \ge 1;$$

Intr-adevar:

$$\mathbf{I.}\ \left.\begin{matrix} \det(X - I_2) = 1 - Tr(X) + \det(X) \\ \det(X + I_2) = 1 + Tr(X) + \det(X) \end{matrix}\right\} \overset{\det(X - I_2) = \det(X + I_2)}{\Rightarrow} Tr(X) = 0;$$

$$\mathbf{II.}\ \text{Din } \mathbf{I};$$

$$\mathbf{III.}\ \text{Fie prin } \mathbf{R.A.}:$$

$$C1: t = 0, \delta \ne 0: \Rightarrow A^2 \overset{HC}{=} -\delta I_2 \Rightarrow A^{2014} = -\delta^{1007} I_2 \overset{\mathbf{II}}{\Rightarrow} \delta = 0 - Contradicte;$$

$$C2: t \ne 0, \delta = 0: \Rightarrow A^2 \overset{HC}{=} tA \overset{inductie}{\Rightarrow} A^n = t^{n-1} A \quad (1) \overset{\mathbf{II}}{\Rightarrow} t^{2014} = 0 \Rightarrow$$
$$t = 0 - Contradicte;$$

$$C3: t \ne 0, \delta \ne 0: \quad A^{2016} = tA^{2015} - \delta A^{2014} \overset{\mathbf{II}}{\Rightarrow} Tr(A^{2015}) = 0 \quad (2);$$

$$A^n = \frac{1}{\delta}\left[tA^{n+1} - A^{n+2}\right] \quad (2) \overset{Tr(A^{2015}) = Tr(A^{2014}) = 0}{\Rightarrow} Tr(A^{2013}) = 0 \overset{(2), Tr(A^{2014}) = 0}{\Rightarrow}$$

$$\ldots\ldots. \overset{recursiv}{\Rightarrow} \ldots. \Rightarrow Tr(I_2) = 0 - Contradicte.$$

$$\mathbf{IV.}\ A^2 \overset{\mathbf{III}, HC}{=} O_2 \Rightarrow A^n = O_2, \forall n \ge 2 \Rightarrow$$

$$\det(A^n - I_2) = \det(A^n + I_2), \forall n \ge 2 \overset{\mathbf{I,III}}{\Rightarrow} \det(A^n - I_2) = \det(A^n + I_2), \forall n \ge 1.$$

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**2.**

Pornind de la o matrice inversabila $A \in M_n(C)$ avand liniile $L_1, L_2, ...., L_n$, construim matricele $B \in M_n(C)$ cu liniile $O, L_2, ...., L_n$ si $C \in M_n(C)$ cu liniile $L_2, ...., L_{n-1}, O$, unde $O$ desemneaza o linie cu toate elementele nule. Fie matricile $D = A^{-1} \cdot B$ si $E = A^{-1} \cdot C$. Aratati ca:

$a) rang(D) = rang(D^2) = .... = rang(D^n)$;

$b) rang(E) > rang(E^2) > .... > rang(E^n)$.

**Solutie.** Consideram matricile $P, Q \in \mathcal{M}_n(\mathbb{C})$,

$$P = \begin{pmatrix} 0 & 0 & 0 & .... & 0 \\ 0 & 1 & 0 & .... & 0 \\ 0 & 0 & 1 & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & 0 & .... & 1 \end{pmatrix}, Q = \begin{pmatrix} 0 & 1 & 0 & .... & 0 \\ 0 & 0 & 1 & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & 0 & .... & 1 \\ 0 & 0 & 0 & .... & 0 \end{pmatrix}$$ si folosim

urmatoarea schema de rezolvare:

**I.** $B = PA$; **II.** $P^k = P, \forall k \geq 1$;

**III.** $rang(D) = rang(D^2) = .... = rang(D^n) = n-1$;

**IV.** $C = QA$; **V.** $Q^k = \left( \begin{array}{c|c} O_{n-k,k} & I_{n-k} \\ \hline \multicolumn{2}{c}{O_{k,n}} \end{array} \right)$;

**VI.** $rang(E) > rang(E^2) > .... > rang(E^n)$;

Intr-adevar:

**I, II.** Din regula de inmultirea matricilor;

**III.** $D = A^{-1}B \overset{\text{I}}{=} A^{-1}PA \Rightarrow D^k = A^{-1}P^kA \overset{\text{II}}{=} A^{-1}PA = D \Rightarrow$

$rang(D^k) = rang(D) = rang(P) = n-1, \forall k = \overline{1,n}$;

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**IV, V.** Din regula de inmultirea matricilor;

$$\textbf{VI. } E = A^{-1}C \overset{ip,\textbf{IV}}{=} A^{-1}QA \Rightarrow E^k = A^{-1}Q^kA \Rightarrow$$

$$rang\left(E^k\right) = rang\left(Q^k\right) \overset{\textbf{V}}{=} n-k \Rightarrow rang(E) > rang\left(E^2\right) > .... > rang\left(E^n\right)$$

**3.**

Fie $a \in \mathbb{R}$. Consideram o functie $f:(0,\infty)\to(0,\infty)$. Aratati ca urmatoarele afirmatii sunt echivalente:

$$(i)\ \lim_{x\to\infty}\frac{f(x)}{x^{a+\varepsilon}} = 0\ si\ \lim_{x\to\infty}\frac{f(x)}{x^{a-\varepsilon}} = \infty, \text{ pentru orice } \varepsilon > 0;$$

$$(ii)\ \lim_{x\to\infty}\frac{\ln f(x)}{\ln x} = a.$$

**Solutie.** Folosim urmatoarea schema de rezolvare:

$"(i)\Rightarrow(ii)"$:

$$\textbf{I. } \varepsilon > 0 \Rightarrow \exists\delta(\varepsilon) > 0\ a.i.\ a-\varepsilon < \frac{\ln f(x)}{\ln x} < a+\varepsilon,\ \forall x\in\left(\delta(\varepsilon),\infty\right);$$

$$\textbf{II. } \lim_{x\to\infty}\frac{\ln f(x)}{\ln x} = a;$$

$"(ii)\Rightarrow(i)"$:

$$\textbf{III. } \varepsilon > 0 \Rightarrow \exists\delta(\varepsilon) > 0\ a.i.\ 0 < \frac{f(x)}{x^{a+\varepsilon}} < \frac{1}{x^{\frac{\varepsilon}{2}}}\ si\ \frac{f(x)}{x^{a-\varepsilon}} > x^{\frac{\varepsilon}{2}},\ \forall x\in\left(\delta(\varepsilon),\infty\right);$$

$$\textbf{IV. } \lim_{x\to\infty}\frac{f(x)}{x^{a+\varepsilon}} = 0\ si\ \lim_{x\to\infty}\frac{f(x)}{x^{a-\varepsilon}} = \infty;$$

Intr-adevar:

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**I.**
$$\left.\begin{array}{l}\lim\limits_{x\to\infty}\dfrac{f(x)}{x^{a-\varepsilon}}=\infty\Rightarrow\exists\delta_1(\varepsilon)>0\ a.i.\ \dfrac{f(x)}{x^{a-\varepsilon}}>1,\ \forall x\in(\delta_1(\varepsilon),\infty)\\ \lim\limits_{x\to\infty}\dfrac{f(x)}{x^{a+\varepsilon}}=0\Rightarrow\exists\delta_2(\varepsilon)>0\ a.i.\ \dfrac{f(x)}{x^{a+\varepsilon}}<1,\ \forall x\in(\delta_2(\varepsilon),\infty)\end{array}\right\}$$

$$\overset{\delta(\varepsilon)=\max\{\delta_1(\varepsilon),\delta_2(\varepsilon)\}}{\Rightarrow}\quad a-\varepsilon<\frac{\ln f(x)}{\ln x}<a+\varepsilon,\ \forall x\in(\delta(\varepsilon),\infty);$$

**II.** Din **I**;

**III.**
$$\left.\begin{array}{l}\lim\limits_{x\to\infty}\dfrac{\ln f(x)}{\ln x}=a\Rightarrow\exists\delta_1(\varepsilon)>0\ a.i.\ a-\dfrac{\varepsilon}{2}<\dfrac{\ln f(x)}{\ln x}\Leftrightarrow x^{\frac{\varepsilon}{2}}<\dfrac{f(x)}{x^{a-\varepsilon}},\ \forall x>\delta_1(\varepsilon)\\ \lim\limits_{x\to\infty}\dfrac{\ln f(x)}{\ln x}=a\Rightarrow\exists\delta_2(\varepsilon)>0\ a.i.\ \dfrac{\ln f(x)}{\ln x}<a+\dfrac{\varepsilon}{2}\Leftrightarrow\dfrac{f(x)}{x^{a+\varepsilon}}<\dfrac{1}{x^{\frac{\varepsilon}{2}}},\ \forall x>\delta_1(\varepsilon)\end{array}\right\}\Rightarrow$$

$$\delta(\varepsilon)=\max\{\delta_1(\varepsilon),\delta_2(\varepsilon)\};$$

**IV.** Din **III** $\Rightarrow 0\le\lim\limits_{x\to\infty}\dfrac{f(x)}{x^{a+\varepsilon}}\le\lim\limits_{x\to\infty}\dfrac{1}{x^{\frac{\varepsilon}{2}}}=0\ si\ \lim\limits_{x\to\infty}\dfrac{f(x)}{x^{a-\varepsilon}}\ge\lim\limits_{x\to\infty}x^{\frac{\varepsilon}{2}}=\infty.$

**4.**

Determinati functiile $f:\mathbb{R}\to\mathbb{R}$ cu proprietatea ca $f^2$ este derivabila pe $\mathbb{R}$ si $\left(f^2\right)'=f$.

**Solutie.**

Observam ca functia identic nula verifica ipoteza, incat putem presupune $f\neq 0$.

Vom considera multimea $A=\{x\in R \mid f(x)=0\}$, pentru care notam $m=\inf(A)\in\overline{R}$, $M=\sup(A)\in\overline{R}$ (se va arata ca multimea $A$ este nevida) si folosim urmatoarea schema de rezolvare:

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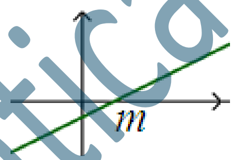
**I.** $f$ este de forma $f(x)=\frac{x}{2}+c$ pe orice interval pe care nu se anuleaza;

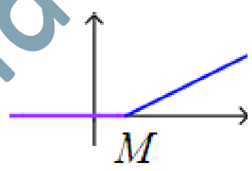
**II.** $A\neq\emptyset$; **III.** $f(m)=0$ (pentru $m\neq-\infty$) si $f(M)=0$ (pentru $M\neq\infty$);

**IV.** Daca $a,b\in A,\ a<b$, atunci $f(x)=0,\ \forall x\in(a,b)$;

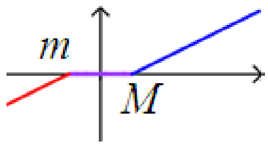
**V.** Daca $m=M\in R$, atunci $f(x)=\frac{x-m}{2},\ \forall x\in R$;



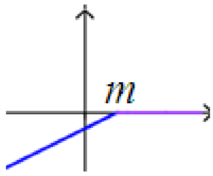

**VI.** Daca $-\infty=m<M<\infty$, atunci $f(x)=\begin{cases}0,\ x\leq M\\ \frac{x-M}{2},\ x>M\end{cases}$;




**VII.** Daca $-\infty<m<M<\infty$, atunci $f(x)=\begin{cases}\frac{x-m}{2},\ x<m\\ 0,\ x\in[m,M]\\ \frac{x-M}{2},\ x>M\end{cases}$




**VIII.** Daca $-\infty<m<M=\infty$, atunci $f(x)=\begin{cases}\frac{x-m}{2},\ x<m\\ 0,\ x\geq m\end{cases}$;




Intr-adevar:

**I.** Fie $I\subset R$ interval si $f(x)\neq 0,\ \forall x\in I \overset{\substack{f=(f^2)'\\ \text{are PD pe } I}}{\Rightarrow} f$ semn constant pe $I\Rightarrow$

$f(x)=\sqrt{f^2(x)}$ sau $f(x)=-\sqrt{f^2(x)}\Rightarrow f$ derivabila pe $I\Rightarrow$

$2f(x)f'(x)=f(x)\overset{f(x)\neq 0}{\Rightarrow} f'(x)=\frac{1}{2}\Rightarrow f(x)=\frac{x}{2}+c,\forall x\in I;$

**II.** Fie prin **R.A.** $A=\emptyset\overset{\mathbf{I}}{\Rightarrow} f(x)=\frac{x}{2}+c,\ \forall x\in R\Rightarrow f(-2c)=0-$ *Contradicte*;

**III.** $C1: m\in A\Rightarrow f(m)=0;\quad C2: m\notin A\Rightarrow\exists(x_n)\subset A$ a.i. $x_n\to m\Rightarrow$

$\lim\limits_{n\to\infty} f^2(x_n)\overset{x_n\in A}{=}\lim\limits_{n\to\infty}0=0\overset{f^2\text{ continua}}{=}f^2(m)\Rightarrow f(m)=0$ si analog pentru $M$;

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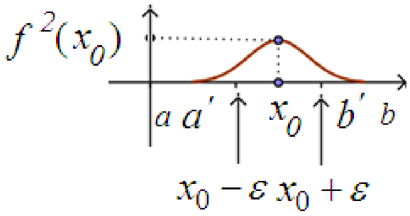
**IV.** Fie prin **R.A.** $x_0 \in (a,b)$ *a.i.* $f(x_0) \neq 0 \overset{f^2\ continua}{\Rightarrow} \exists \varepsilon > 0$ *a.i.* $a < x_0 - \varepsilon < x_0 + \varepsilon < b$ *si*




$f^2(x) > 0, \forall x \in [x_0 - \varepsilon, x_0 + \varepsilon] \Rightarrow$
$f(x) \neq 0, \forall x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ (1).

Fie $a' = \sup\{ x \in [a, x_0 - \varepsilon) \mid f(x) = 0 \}$ *si* $b' = \inf\{ x \in (x_0 + \varepsilon, b] \mid f(x) = 0 \} \overset{(1)}{\Rightarrow}$
$f(x) \neq 0, \forall x \in (a', b') \overset{\mathbf{I}}{\Rightarrow} f(x) = \frac{x}{2} + c, \forall x \in (a', b')$ (2).

Pe de alta parte, asemanator ca la etapa **III**, se arata ca $f(a') = f(b') = 0$ – *Contradictie cu* (2).

**V.** Avem $f(x) \overset{\mathbf{I}}{=} \begin{cases} \frac{x}{2} + c_1, & x < m \\ \frac{x}{2} + c_2, & x > m \end{cases} \overset{\substack{\mathbf{III} \\ f^2\ continua}}{\Rightarrow} f(x) = \frac{x - m}{2}, \forall x \in \mathbb{R};$

**VI, VII, VIII.** Rezulta tinand cont de etapele **I**, **III** si **IV**.