OLIMPIADA NATIONALA DE MATEMATICA
ETAPA FINALA - 2015

1.

Sa se determine functiile derivabile $f:\mathbb{R}\to\mathbb{R}$ care verifica simultan conditiile:

i. $f'(x)=0$ pentru orice $x\in\mathbb{Z}$;

ii. pentru $x\in\mathbb{R}$, daca $f'(x)=0$, atunci $f(x)=0$.

Solutie.

Aratam ca functia nula este singura functie care indeplineste conditiile problemei.

Fie prin **R.A.** o functie $f:\mathbb{R}\to\mathbb{R}$ care indeplineste conditia problemei si $x_0\in\mathbb{R}$ astfel incat $f(x_0)>0$ (de exemplu). Folosim urmatoarea schema de rezolvare:

I. $k=[x_0]<x_0<[x_0]+1=k+1;$

II. f isi atinge maximul pe intervalul $[k,k+1]$ in $x_M\in(k,k+1);$

III. *Contradictia*;

Intr-adevar:

I.,II. Din $f(k),\ f(k+1)=0<f(x_0);$

III. $f'(x_M)\overset{TF,\mathbf{II}}{=}0\overset{ip}{\Rightarrow}f(x_0)=0-Contradictie.$

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2.

Fie $A \in \mathcal{M}_5(\mathbb{C})$ o matrice cu $tr(A) = 0$ si cu proprietatea ca $A - I_5$ este inversabila. Sa se arate ca $A^5 \neq I_5$.

Solutie.

Fie $\varepsilon = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ si $V = \{\lambda \in C \mid \det(A - \lambda I_5) = 0\}$. Presupunem prin **R.A.** ca $A^5 = I_5$ si folosim urmatoarea schema de rezolvare:

$$\textbf{I.}\, V \subset \{\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4\}; \quad \textbf{II.}\, \exists a,b,c,d \in \mathbb{N}\ a.i. \begin{cases} a+b+c+d=5 \\ a+b\varepsilon+c\varepsilon^2+d\varepsilon^3=0 \end{cases};$$

$$\textbf{III.} - Contradictia;$$

Intr-adevar:

$$\textbf{I.} \left.\begin{array}{l} A^5 = I_5 \Rightarrow \text{singura valoare proprie a lui } A^5 \text{ este } \mu = 1 \overset{\lambda \in V}{\Rightarrow} \lambda^5 = 1 \\ A - I_5\ inversabila \Rightarrow \lambda = 1\ nu\ este \text{ valoare proprie a matricei } A \end{array}\right\} \Rightarrow$$

$$V \subset \{\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4\};$$

$$\textbf{II.}\, tr(A) = 0 \overset{tr(A) = \sum_{\lambda \in V} \lambda}{\Rightarrow} \exists a,b,c,d \in \mathbb{N}\ a.i. \begin{cases} a+b+c+d=5 \\ a\varepsilon+b\varepsilon^2+c\varepsilon^3+d\varepsilon^4=0 \end{cases} \Rightarrow$$

$$\exists a,b,c,d \in \mathbb{N}\ a.i. \begin{cases} a+b+c+d=5 \\ a+b\varepsilon+c\varepsilon^2+d\varepsilon^3=0 \end{cases};$$

$$\textbf{III.}\, a+b\varepsilon+c\varepsilon^2+d\varepsilon^3 \overset{\textbf{II}}{=} 0 \Leftrightarrow \begin{cases} a+b\cos\frac{2\pi}{5}+c\cos\frac{4\pi}{5}+d\cos\frac{6\pi}{5}=0 \\ b\sin\frac{2\pi}{5}+c\sin\frac{4\pi}{5}+d\sin\frac{6\pi}{5}=0 \end{cases} \Leftrightarrow$$

$$\begin{cases} a+b\cos\frac{2\pi}{5}-(c+d)\cos\frac{\pi}{5}=0 \\ \sin\frac{\pi}{5}\left[2b\cos\frac{\pi}{5}+c-d\right]=0 \end{cases} \Leftrightarrow \begin{cases} a+b\frac{\sqrt{5}-1}{4}-(c+d)\frac{\sqrt{5}+1}{4}=0 \\ b\frac{\sqrt{5}+1}{2}+c-d=0 \end{cases} \Rightarrow$$

$$a=b=c=d=0 \Rightarrow a+b+c+d=0 - Contradictie\ \text{cu}\ \textbf{II}.$$

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3.

Fie $a \geq 0$ si $(x_n)_{n\geq 1}$ un sir de numere reale. Sa se arate ca daca sirul $\left(\frac{x_1+....+x_n}{n^a}\right)_{n\geq 1}$ este marginit, atunci sirul $(y_n)_{n\geq 1}$, definit prin $y_n = \frac{x_1}{1^b}+\frac{x_2}{2^b}+....+\frac{x_n}{n^b}$, este convergent pentru orice $b > a$.

Solutie.

Notam cu $S_n = x_1 + ... + x_n$, consideram functia $f:(0,\infty) \to (0,\infty)$, $f(x) = \frac{1}{x^\alpha}, \alpha > 0$ si folosim urmatoarea schema de rezolvare:

$\textbf{I.}\ \exists c > 0\ a.i. |S_n| < c \cdot n^a, \forall n \geq 1;\quad \textbf{II.}\ \frac{b}{(k+1)^{b+1}} < \frac{1}{k^b} - \frac{1}{(k+1)^b} < \frac{b}{k^{b+1}}, \forall k \geq 1;$

$\textbf{III.}\ \frac{b-a}{(k+1)^{b-a+1}} < \frac{1}{k^{b-a}} - \frac{1}{(k+1)^{b-a}} < \frac{b-a}{k^{b-a+1}}\ \forall k \geq 1;$

$\textbf{IV.}\ y_{n+p} - y_n = \frac{S_{n+p}}{(n+p)^b} - \frac{S_n}{(n+1)^b} + S_{n+1}\left[\frac{1}{(n+1)^b} - \frac{1}{(n+2)^b}\right] + ...+ S_{n+p-1}\left[\frac{1}{(n+p-1)^b} - \frac{1}{(n+p)^b}\right], \forall n, p \geq 1;$

$\textbf{V.}\ |y_{n+p} - y_n| \leq c\left(2 + \frac{1}{b-a}\right)\frac{1}{n^{b-a}}, \forall n, p \geq 1;$

$\textbf{VI.}\ (y_n)$ sir Cauchy $\Rightarrow (y_n)$ convergent;

Intr-adevar:

I. Din $\left(\frac{S_n}{n^a}\right)$ marginit; **II, III.** Aplicand teorema lui Lagrange functiei f pe intervalul $[k, k+1]$, obtinem $\frac{1}{(k+1)^\alpha} - \frac{1}{k^\alpha} = -\frac{\alpha}{c^{\alpha+1}} \Leftrightarrow \frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} = \frac{\alpha}{c^{\alpha+1}} \overset{c\in(k,k+1)}{\Rightarrow}$

$\frac{\alpha}{(k+1)^{\alpha+1}} < \frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} < \frac{\alpha}{k^{\alpha+1}}$. Alegem apoi $\alpha = b$, respectiv $\alpha = b - a$;

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IV. $y_{n+p}-y_n=\frac{x_{n+1}}{(n+1)^b}+\frac{x_{n+2}}{(n+2)^b}+....+\frac{x_{n+p}}{(n+p)^b}=$

$$\frac{S_{n+1}-S_n}{(n+1)^b}+\frac{S_{n+2}-S_{n+1}}{(n+2)^b}+....+\frac{S_{n+p}-S_{n+p-1}}{(n+p)^b}=$$

$$\frac{S_{n+p}}{(n+p)^b}-\frac{S_n}{(n+1)^b}+S_{n+1}\left[\frac{1}{(n+1)^b}-\frac{1}{(n+2)^b}\right]+...+S_{n+p-1}\left[\frac{1}{(n+p-1)^b}-\frac{1}{(n+p)^b}\right];$$

V. $\left|y_{n+p}-y_n\right|\overset{\text{I,II}}{\leq} c\left(\frac{(n+p)^a}{(n+p)^b}+\frac{n^a}{(n+1)^b}+\frac{b(n+1)^a}{(n+1)^{b+1}}+....+\frac{b(n+p-1)^a}{(n+p-1)^b}\right)=$

$$c\left[\frac{1}{(n+p)^{b-a}}+\frac{1}{(n+1)^{b-a}}+\frac{b}{(n+1)^{b-a+1}}+....+\frac{b}{(n+p-1)^{b-a+1}}\right]\overset{\text{III}}{\leq}$$

$$c\left[\frac{2}{n^{b-a}}+\frac{b}{b-a}\left(\frac{1}{n^{b-a}}-\frac{1}{(n+1)^{b-a}}+....+\frac{1}{(n+p-2)^{b-a}}-\frac{1}{(n+p-1)^{b-a}}\right)\right]\leq$$

$$c\left[\frac{2}{n^{b-a}}+\frac{b}{b-a}\left(\frac{1}{n^{b-a}}-\frac{1}{(n+p-1)^{b-a}}\right)\right]\leq c\left(\frac{2}{n^{b-a}}+\frac{b}{b-a}\frac{1}{n^{b-a}}\right)=$$

$\left(2+\frac{1}{b-a}\right)\frac{1}{n^{b-a}}$; **VI.** Din **V** si $\lim\limits_{n\to\infty} c\left(2+\frac{1}{b-a}\right)\frac{1}{n^{b-a}}=0.$

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4.

Matricile $A \in \mathcal{M}_{m,n}(\mathbb{C})$ si $B \in \mathcal{M}_{n,m}(\mathbb{C})$, unde $m \geq n \geq 2$ au proprietatea ca exista $k \in \mathbb{N}^*$ si $a_0, a_1, ..., a_k \in \mathbb{C}$ astfel incat:

$$a_k(AB)^k + a_{k-1}(AB)^{k-1} + ... + a_1(AB) + a_0 I_m = O_m \text{ si}$$
$$a_k(BA)^k + a_{k-1}(BA)^{k-1} + ... + a_1(BA) + a_0 I_n \neq O_n.$$

Demonstrati ca $a_0 = 0$.

Solutie. Presupunem prin *reducere la absurd* ca $a_0 \neq 0$ si folosim urmatoarea schema de rezolvare:

I. $rang(AB) = m$; **II.** $m = n$; **III.** A, B inversabile; **IV.** *Contradictia*;

Intr-adevar:

$$\textbf{I. } I_m = -\frac{1}{a_0}\left[a_k(AB)^k + ... + a_1(AB)\right] \Rightarrow AB \text{ inversabila} \Rightarrow rang(AB) = m;$$

$$\textbf{II. } m \overset{\textbf{I}}{=} rang(AB) \leq \min\{rang(A), rang(B)\} \leq n \overset{ip}{\Rightarrow} m = n; \quad \textbf{III. } \text{Din } \textbf{I} \text{ si } \textbf{II};$$

$$\textbf{IV. } a_k(AB)^k + a_{k-1}(AB)^{k-1} + ... + a_1(AB) + a_0 I_m = O_m \overset{\substack{immultire \\ st.\ B,\ dr.\ A}}{\Longrightarrow}$$

$$a_k(BA)^{k+1} + a_{k-1}(BA)^k + ... + a_1(BA)^2 + a_0(BA) = O_n \overset{\substack{inmultire \\ (BA)^{-1}}}{\Longrightarrow}$$
$$a_k(BA)^k + a_{k-1}(BA)^{k-1} + ... + a_1(BA) + a_0 I_n = O_n - \text{Contradictie.}$$